THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050A Tutorial 6

1. Show that $\lim_{x\to 0} \cos(1/x)$ does not exist.

2. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ and $x_0, y_0, l \in \mathbb{R}$. If

   (i) $\lim_{x\to x_0} g(x) = y_0$ and $\lim_{y\to y_0} f(y) = l$; and

   (ii) there exists $\delta > 0$ such that $g(x) \neq y_0$ for any x satisfying $0 < |x - x_0| < \delta$,

   show that $\lim_{x\to x_0} f(g(x)) = l$. Can we drop condition (ii)?

3. Prove the **Squeeze Theorem**: Let $A \subseteq \mathbb{R}$, let $f, g, h : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If

   $$f(x) \leq g(x) \leq h(x) \qquad \text{for all } x \in A, x \neq c,$$

   and

   $$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x),$$

   show that $\lim_{x\to c} g(x) = L$.

4. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and c be a cluster point of both of the sets $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Show that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^+} f(x) = L = \lim_{x\to c^-} f(x)$.

5. (a) State the definition of limits at infinity.

   (b) Evaluate $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x} - x}{\sqrt{x} + x}$ (if exist) by definition.

6. Let $f : (0, \infty) \to \mathbb{R}$. Prove that $\lim_{x\to\infty} f(x) = L$ if and only if $\lim_{x\to 0^+} f(1/x) = L$.